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# Persistent Saddle Connections in a Class of Reaction–Diffusion Equations

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## 1. INTRODUCTION



$$u_t = \Delta u + f(u), \qquad t > 0, \quad x \in \Omega, \tag{1.1}$$

$$u = 0, \qquad t > 0, \quad x \in \partial\Omega, \tag{1.2}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $f$ is a $C^1$ function on $\bar{\Omega} \times \mathbb{R}$. Problem (1.1), (1.2) defines a local semiflow on an appropriate Banach space, for example, the Sobolev space

$$X^{1/2} := W_0^{1,p}(\Omega),$$

with a $p > N$ (see [He1; Am; Lu; Da-K]). The semiflow is gradient-like: the energy functional

$$\varphi \mapsto \int_\Omega \left(\tfrac{1}{2}\,|\nabla\varphi(x)|^2 - F(\varphi(x))\right) dx,$$

where $F = \int f$, decreases along any nonconstant trajectory. This in particular implies that any bounded solution approaches as $t \to \infty$ a connected set of equilibria (see [Ha; He1]). If all the equilibria are isolated, then any bounded trajectory converges to a single equilibrium and any trajectory that is defined and bounded for all $t \in \mathbb{R}$ is a heteroclinic trajectory between

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two equilibria. In this situation, if (1.1), (1.2) possesses a compact global attractor (in the sense of [Ha]), then the attractor consists entirely of heteroclinic trajectories.

We say that (1.1), (1.2) has the Morse–Smale property if the following two conditions are satisfied:

(P1) each equilibrium $\varphi$ of (1.1), (1.2) is hyperbolic, that is, the linearization at $\varphi$ does not have zero as an eigenvalue,

(P2) for any two equilibria $\varphi$, $\psi$, the unstable manifold of $\varphi$ intersects the stable manifold of $\psi$ transversally:

$$W^u(\varphi) \pitchfork W^s(\psi).$$

The Morse–Smale property is closely related to structural stability. If an appropriate dissipativity condition is assumed, so that (1.1), (1.2) has a global compact attractor $\mathscr{A}$, then (P1), (P2) guarantee that the flow on the attractor does not change qualitatively under small $C^1$ perturbations of (1.1), (1.2). More specifically, any small $C^1$ dissipative perturbation of the semiflow of (1.1), (1.2) has an attractor $\tilde{\mathscr{A}}$ near $\mathscr{A}$ and the flow on $\tilde{\mathscr{A}}$ is conjugate to the flow of (1.1), (1.2) on $\mathscr{A}$ (see [Ha-M-O]). For a general background on the Morse–Smale property we refer the reader to [Pa-M; Sma]). Note that additional conditions are involved in the definition if the flow is not gradient-like.

It has been known for more than a decade (see [He1, An]) that problems (1.1), (1.2) in one space dimension enjoy the remarkable property that (P2) is automatically satisfied for any two hyperbolic equilibria. This is true for a more general form of 1D equations; for example, one can consider nonlinearities depending on $x$ and $u_x$: $f = f(x, u, u_x)$, and allow for general separated boundary conditions (see also [Che-C-H] for an improvement and extension of these results).

In contrast, (P2) is no longer automatic if $N > 1$. In [Po1; Po2] examples of equations with $f = f(x, u)$ are found such that (P2) fails to hold for some hyperbolic equilibria. As we show in this paper, nontransversal intersections of stable and unstable manifolds of equilibria can occur for spatially independent nonlinearities $f = f(u)$ as well.

With (P2) not being satisfied always, one asks naturally "how often" it is satisfied. In view of the Kupka–Smale theorem for finite dimensional vector fields (cf. [Pa-M]), one would expect that (P2) holds generically. This is likely to be true in some sense, however, one has to be a little careful about the statement. We prove in this paper that (P2) is not a generic property of the nonlinearity $f = f(u)$ in any reasonable $C^1$ topology. The precise formulation of our result is as follows:

THEOREM 1. *There exists a smooth convex domain $\Omega \subset \mathbb{R}^2$ and a quadratic function $f_0(u) = \mu u - u^2$ with the following property. For any $C^1$ real function $f$ with*

$$\|f - f_0\|_{C^1[-1,1]} := \sup_{u \in [-1,1]} (|f_0(u) - f(u)| + |f_0'(u) - f'(u)|)$$

*sufficiently small,* (1.1), (1.2) *has two hyperbolic equilibria $\varphi$, $\psi$ such that $W^u(\varphi)$ and $W^s(\psi)$ intersect nontransversally.*

This result shows that the class of $x$-independent nonlinearities is too small for (P2) to be generic; it does not provide for a sufficiently general perturbation to break a nontransverse intersection once it has occured. Note that for a more general class of equations, namely (1.1), (1.2) with $f = f(x, u)$, (P1), (P2) is a generic property of $f$ in reasonable $C^k$ topologies for any $k \geqslant 1$ (see [Br-P]).

It is quite natural to take $\Omega$ into consideration when generic properties of (1.1), (1.2) are discussed. On a fixed domain, symmetries of $\Omega$ may impose a special equivariance structure on (1.1), (1.2), which cannot be removed by perturbation of $f$ in the spatially homogeneous class. A (discrete) symmetry indeed plays an important role in our construction below. A perturbation of $\Omega$ destroys the symmetry, which gives (P2) a chance to be generic in a space of domains. Note that property (P1), the hyperbolicity of equilibria, is generic under the domain perturbation (see [He2]).

Theorem 1 is a direct consequence of Theorems 2.1 and 3.1, proved in Sections 2, 3. More specifically, in Section 2 we consider a bifurcation problem for (1.1), (1.2) with varying domain $\Omega$. Under certain conditions (involving in particular a reflectional symmetry of the domain), a heteroclinic orbit is found near a bifurcation point. This heteroclinic orbit is transversal in the space of symmetric functions (thus persists under perturbations of $f = f(u)$) but is not transversal in the full state space. The latter is shown by calculating the Morse indices of the bifurcating equilibria. The hypotheses imply that the Morse indices are equal, and hence the heteroclinic connection cannot be transversal.

In Section 3, we find domains that fit the setup of Section 2. Some, but not all, of the hypotheses needed are satisfied on the disk in $\mathbb{R}^2$. By a suitable perturbation of the disk, we achieve that all the hypotheses are satisfied. We believe that the calculations of derivatives of eigenvalues and eigenfunctions carried out in Section 3 might be of independent interest.

Although we restrict our attention to $\mathbb{R}^2$, the method can be used in higher dimensions as well. It can also be adapted for other classes of equations with reflectional symmetries. For example, equations with nonlinearity $f = f(u, u_x, u_y)$ even in the last variable can be treated in a similar way.

In the remaining part of the Introduction we recall the definitions of a few basic concepts considered in this paper. An equilibrium $\varphi$ of (1.1), (1.2) is hyperbolic if the operator $\Delta + f'(\varphi(x))$ under the Dirichlet boundary condition does not have zero (thus any other purely imaginary number) as an eigenvalue. The unstable manifold, $W^u(\varphi)$, of such an equilibrium is the set all initial data $u_0 \in X^{1/2}$ with the property that there exists a solution $u(\cdot, t)$ of (1.1), (1.2) defined for all $t \in (-\infty, 0]$ such that $u(\cdot, 0) = u_0$ and $u(\cdot, t) \to \varphi$ as $t \to -\infty$. The stable manifold, $W^s(\varphi)$, is defined analogously with solutions on $[0, \infty)$ considered instead. Both $W^u(\varphi)$ and $W^s(\varphi)$ are $C^1$ submanifolds of $X^{1/2}$ with $\dim W^u(\varphi) = \operatorname{codim} W^s(\varphi)$ given by the dimension of the eigenspace of $\Delta + f'(\varphi(x))$ corresponding to the set of all positive eigenvalues (see [He1]). For two hyperbolic equilibria $\varphi$ and $\psi$, the manifolds $W^u(\varphi)$ and $W^s(\psi)$ intersect transversally if, at any point of intersection, the tangent space of $W^u(\varphi)$ contains a complement in $X^{1/2}$ to the tangent space of $W^s(\psi)$. Note that in autonomous equations this is possible only if

$$\operatorname{codim} W^s(\psi) < \dim W^u(\varphi).$$

## 2. A BIFURCATION INDUCED BY DOMAIN PERTURBATION

In this section we show that under certain conditions a persistent saddle connection for (1.1), (1.2) is obtained as a result of bifurcation when perturbing the domain. We start by putting the domain perturbation problem in a suitable functional-analytic framework. We mostly follow [He2] in doing that.

Fix a smooth domain $\Omega_0 \subset \mathbb{R}^2$. For a $C^2$ imbedding $h: \bar{\Omega}_0 \to \mathbb{R}^2$ we denote

$$\Omega_h := h(\Omega_0).$$

Given a function $u: \Omega_h \to \mathbb{R}$, we define $h^*u$, the pull-back of $u$, by

$$h^*u(x) = u(h(x)) \qquad (x \in \Omega_0).$$

Note that $u \mapsto h^*u$ maps $W^{i,p}(\Omega_h)$ onto $W^{i,p}(\Omega_0)$ $(i = 0, 1, 2)$, $W_0^{1,p}(\Omega_h)$ onto $W_0^{1,p}(\Omega_0)$, $C^2(\bar{\Omega}_h)$ onto $C^2(\bar{\Omega}_0)$, and is actually an isomorphism between these Banach spaces. The inverse operator is given by

$$(h^*)^{-1} v(y) = v(h^{-1}(y)).$$

We shall use the same symbol $h^*$ for the map $u \mapsto h^*u$ acting between various spaces. This should cause no confusion.

Fix a $p > 2$. For a domain $\Omega$ let $\Delta_\Omega$ denote the $L^p(\Omega)$ realization of the Laplacian under Dirichlet boundary condition:

$$D(\Delta_\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$
$$(\Delta_\Omega u)(x) = \Delta u(x) \qquad (x \in \Omega).$$

We consider the problem:

$$z_t = L_h z + \hat{f}(z), \tag{2.1}$$

where

$$L_h = h^* \Delta_{\Omega_h} (h^*)^{-1}$$

and $\hat{f}$: $W_0^{1,p}(\Omega_0) \to L^p(\Omega_0)$ is the Nemitskii operator of the function $f$:

$$\hat{f}(z)(x) = f(z(x)).$$

This problem can be considered in the context of the analytic semigroup generated by $L_h$ on

$$X := L^p(\Omega_0).$$

The analytic semigroup generated by $L_h$ on $X$ is given by

$$h^* e^{t\Delta_{\Omega_h}} (h^*)^{-1}, \qquad t \geqslant 0,$$

where $e^{t\Delta_{\Omega_h}}$, $t \geqslant 0$, is the semigroup generated by $\Delta_{\Omega_h}$. The space

$$X^{1/2} := W_0^{1,p}(\Omega_0)$$

(with the standard norm replaced by an equivalent one) is a fractional power space of $L_h$ (see [He1; Lu]). Since $\hat{f}$: $X^{1/2} \to X$ is of class $C^1$, (2.1) is well posed on $X^{1/2}$. We denote by $z(\cdot, t, z_0, h)$ (by $z(\cdot, t, z_0)$, if there is no danger of confusion) the maximally defined solution of (2.1) with the initial condition $z_0 \in W_0^{1,p}(\Omega_0)$. This way we obtain a local semiflow on $X^{1/2}$. By standard regularity results each solution is a classical solution and, in particular, the stationary solutions are contained in

$$X^1 := W^{2,p}(\Omega_0) \cap W_0^{1,p}(\Omega_0).$$

Note that the solutions of (2.1) are in one-to-one correspondence, via $h^*$, with the solutions of (1.1), (1.2) where $\Omega = \Omega_h$ (we have just transformed the coordinates, so that we can work on a fixed domain.) It is obvious that

$h^*$ yields a one-to-one correspondence between equilibria, their stable and unstable manifolds and preserves the property of these manifolds being transversal or nontransversal. Thus, finding an Eq. (2.1) with a non-transversal intersections will at the same time provide an example of (1.1), (1.2) (with $\Omega = \Omega_h$) with the same property.

From now on we shall only consider domains that are symmetric under the reflection about the $x_1$ axis. More specifically, let

$$J(x_1, x_2) = (x_1, -x_2). \tag{2.2}$$

Assume that

(S1) $J\Omega_0 = \Omega_0$,

(S2) $Jh = hJ$.

These conditions imply

$$J\Omega_h = \Omega_h.$$

Let $X_e$, respectively $X_o$, denote the subspace of $X$ consisting of all functions that are even, respectively odd in $x_2$. Similarly, for the spaces introduced above, we denote

$$X_e^i = X^i \cap X_e, \qquad X_o^i = X^i \cap X_o \qquad (i = 1, \tfrac{1}{2}).$$

Conditions (S1), (S2) imply that $X_e^{1/2}$ (and similarly $X_o^{1/2}$) is invariant under the flow of (2.1): if $z_0 \in X_e^{1/2}$ then $z(\cdot, t, z_0) \in X_e^{1/2}$ as long as the solution exists. Furthermore, if $\varphi$ is an equilibrium of (2.1) contained in $X_e^{1/2}$, then the linearization,

$$A = L_h + \hat{f}'(\varphi),$$

has $X_o$, $X_e$ as invariant subspaces. In particular,

$$\sigma(A) = \sigma(A|_{X_o}) \cup \sigma(A|_{X_e}).$$

(Here $A$ is understood as a closed unbounded operator on $X$.) For any such equilibrium we define $m_e(\varphi)$ to be the number of positive eigenvalues of $A|_{X_e}$, and $m(\varphi)$ to be the number of positive eigenvalues of (the not restricted operator) $A$, in both cases we count with multiplicities. If the equilibrium $\varphi$ is hyperbolic, then

$$m(\varphi) = \dim W^u(\varphi),$$

$$m_e(\varphi) = \dim W_e^u(\varphi) \qquad \text{with} \quad W_e^u(\varphi) = W^u(\varphi) \cap X_e^{1/2},$$

that is, $m(\varphi)$, $m_e(\varphi)$ are the Morse indices of $\varphi$ relative to the flow of (2.1) on $X^{1/2}$ and on $X_e^{1/2}$, respectively.

We next consider a family of domains corresponding to a one-parameter family of diffeomorphisms

$$h_\lambda = I + \lambda\zeta.$$

Here $I$ is the identity on $\mathbb{R}^2$, $\zeta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function and $\lambda \in (-\varepsilon, \varepsilon)$ is a real parameter. We assume that $\varepsilon > 0$ is sufficiently small so that $h_\lambda|_{\bar{\Omega}_0}$ is an imbedding of $\bar{\Omega}_0$ (below $\varepsilon$ may have to made smaller so that other requirements are also satisfied). Abusing notation slightly, we abbreviate

$$\Omega_\lambda = \Omega_{h_\lambda}, \qquad L_\lambda = L_{h_\lambda}. \tag{2.3}$$

We assume the following additional hypotheses:

(S2)′ $J\zeta = \zeta J$ (so that (S2) holds for $h_\lambda$).

(S3) The second eigenvalue, $\mu_2$, of $-L_0$ is a double eigenvalue and

$$\ker(L_0 + \mu_2) = \operatorname{span}\{v, w\} \qquad \text{with} \quad v \in X_e^1, \quad w \in X_o^1.$$

(Note that $L_0 = L_\lambda|_{\lambda=0}$ is the $L^p(\Omega_0)$ realizations of the Laplacian under Dirichlet boundary condition.)

(S4) $d_1 := \int_{\partial\Omega_0} (\partial v/\partial \mathfrak{n})^2 \zeta \cdot \mathfrak{n} > 0 > d_2 := \int_{\partial\Omega_0} (\partial w/\partial \mathfrak{n})^2 \zeta \cdot \mathfrak{n}$,

where $\mathfrak{n}$ is the unit outward normal field on $\partial\Omega_0$.

(S5) $\int_{\Omega_0} v^3 \neq 0$, and, with $d_1$, $d_2$ as in (S4),

$$d_2 - 2\frac{\int_{\Omega_0} vw^2}{\int_{\Omega_0} v^3} d_1 > 0.$$

We prove the existence of a domain $\Omega_0$ and a function $\zeta$ satisfying these conditions in Section 3 (see Theorem 3.1).

THEOREM 2.1. *Let $\Omega_0 \subset \mathbb{R}^2$ be a smooth domain and $\zeta\colon \mathbb{R}^2 \to \mathbb{R}^2$ be smooth function such that the hypotheses* (S1), (S2)′, (S3)–(S5) *are satisfied. Let $f_0$ be any $C^2$-function satisfying the relations*

$$f_0(0) = 0, \qquad f_0'(0) = \mu_2, \qquad f_0''(0) \neq 0 \tag{2.4}$$

*(e.g., $f_0(u) = \mu_2 u \pm u^2$). For any $\lambda$ sufficiently small and positive there exist a $\delta > 0$ and a neighborhood $U$ of 0 in $X_e^{1/2}$ such that if $\Omega = \Omega_\lambda$ and $f$ is any real function with*

$$\|f - f_0\|_{C^1[-1,1]} < \delta,$$

*then the following statements hold true:*

(i) (2.1) *has exactly two equilibria in $U$, say $\varphi$, $\psi$, and both of them are hyperbolic,*

(ii) $m_e(\varphi) = 2$, $m_e(\psi) = 1$,

(iii) *there is a heteroclinic (connecting) orbit in $X_e^{1/2}$ from $\varphi$ to $\psi$,*

(iv) $m(\varphi) = 2 = m(\psi)$.

*In particular,* (iii), (iv) *imply that $W^u(\varphi)$ and $W^s(\psi)$ have a nontransversal intersection.*

*Proof.* The outline of the proof is as follows. We first consider (2.1) with $f = f_0$ fixed. Using a Lyapunov–Schmidt reduction, we analyze the bifurcation of stationary solutions as $\lambda$ passes through 0. Restricting to $X_e$ first, we study a bifurcation from the simple eigenvalue $\mu_2$ in order to find equilibria and calculate their Morse indices relative to $X_e$. Then, using perturbation analysis, we find the full Morse indices. The properties found here will persist for $f \approx f_0$. Employing a local center manifold, we then establish the existence of a connecting orbit, as in (iii), and complete the proof. We remark that a similar scenario was used in [Po1], where the nonlinearity $f = f(x, u)$ involved a varying parameter and the domain was fixed.

We assume the hypotheses of the theorem to be satisfied throughout the section.The hypotheses are not affected if we also assume that the functions $v$, $w$, as in (S4) are normalized in $L^2$:

$$\int_{\Omega_0} v^2 = 1, \qquad \int_{\Omega_0} w^2 = 1.$$

To start the bifurcation analysis, consider the map $F$: $X_e^1 \times (-\varepsilon, \varepsilon) \to X_e$ defined by

$$F(z, \lambda) = L_\lambda z + \hat{f}_0(z). \tag{2.5}$$

Observe that $F$ is of class $C^2$,

$$\ker D_z F(0, 0) = \operatorname{span}\{v\},$$

$$R(D_z F(0, 0)) = R(I - P),$$

where $I$ is the identity on $X_e$ and $P\colon X_e \to \operatorname{span}\{v\}$ is the restriction to $X_e$ of the orthogonal projection of $L^2(\Omega_0)$:

$$Pz = \left(\int_\Omega vz\right) v. \tag{2.6}$$

Using a Lyapunov–Schmidt reduction one obtains that for a neighborhood $U$ of 0 in $X_e$ and a sufficiently small $\varsigma > 0$ the zero set of $F$ is locally given by

$$F^{-1}(0) \cap (U \times (-\varsigma, \varsigma)) = \{(sv + \xi(s, \lambda), \lambda) : \beta(s, \lambda) = 0\}, \tag{2.7}$$

where $\xi$ and $\beta$ are $C^2$-functions defined on $U \times (-\varsigma, \varsigma)$, taking values in $R(I-P)$ and $\mathbb{R}$, respectively, and assuming the zero values at (0, 0). Let us recall briefly how these functions are found (see for example [Smo, Sect. II.13.A, B; Va1, Sect. 6.4.11; Go-S; Cho-H]) for details). Writing $z = sv + y$ $(s \in \mathbb{R},\ y \in R(I-P) \cap X_e^1)$, the equation $F(z, \lambda) = 0$ is equivalent to the following system of equations for $(s, y, \lambda)$:

$$PF(sv + y, \lambda) = 0,$$

$$(I-P)\, F(sv + y, \lambda) = 0.$$

Using the implicit function theorem for $(s, y, \lambda)$ near (0, 0, 0), one solves the second equation for $y$, thus obtaining a function $y = \xi(s, \lambda)$. Substituting in the first equation, one obtains the bifurcation equation

$$PF(sv + \xi(s, \lambda), \lambda) = 0,$$

which is equivalent to the original equation in a neighborhood of (0, 0, 0). Defining $\beta$ by the identity

$$PF(sv + \xi(s, \lambda), \lambda) = \beta(s, \lambda)\, v,$$

one obtains (2.7)

In the forthcoming calculations we shall need formulas for a few derivatives of $\xi$ and $\beta$. They are found by the implicit differentiation (cf. [Go-S])

$$\begin{aligned} \xi_s(0, 0) &= 0, \\ \beta_s(0, 0) &= 0, \\ \beta_{ss}(0, 0)\, v &= PD^2_{zz} F(0, 0)\, v^2, \\ \beta_{s\lambda}(0, 0)\, v &= PD^2_{z\lambda} F(0, 0)\, v. \end{aligned} \tag{2.8}$$

(The last formula is simpler than in the general calculations, due to $D_\lambda F(0, 0) = 0$.) Using (2.5), (2.6), we obtain

$$\beta_{ss}(0, 0) = f_0''(0) \int_{\Omega_0} v^3, \tag{2.9}$$

and

$$\beta_{s\lambda}(0, 0) = \int_{\Omega_0} v \frac{d}{d\lambda} (L_\lambda v) \bigg|_{\lambda = 0}. \tag{2.10}$$

To calculate the latter we use the following lemma.

Lemma 2.2. *One has*

$$\frac{d}{d\lambda} (L_\lambda v) \bigg|_{\lambda = 0} = [\zeta \cdot \nabla, \Delta]\, v,$$

*where* $[\zeta \cdot \nabla, \Delta]$ *is the commutator of* $\zeta \cdot \nabla$ *and* $\Delta$:

$$[\zeta \cdot \nabla, \Delta]\, v = \zeta \cdot \nabla(\Delta v) - \Delta(\zeta \cdot \nabla v).$$

Note that $[\zeta \cdot \nabla, \Delta]$ is a second order operator.

Lemma 2.2 is a particular case of a theorem on general differential operators proved in [He2, Section 2]. (Similar calculations can be found in [Pe].)

In our setting above, $h_\lambda$ is a linear perturbation of the identity, $h_\lambda = I + \lambda\zeta$. For future purposes we remark that the calculation of the derivative is unaffected if $h_\lambda$ involves higher order terms in $\lambda$,

$$h_\lambda(x) = x + \lambda\zeta(x) + o(\lambda), \qquad \text{as} \quad \lambda \to 0,$$

as soon as $(x, \lambda) \to h_\lambda(x)$ is sufficiently smooth. A sufficient (not optimal) regularity is that $\partial_\lambda^j \partial_x^k h_\lambda(x)$ are continuous for $0 \leqslant j \leqslant 1$, $0 \leqslant k \leqslant 3$.

Using Lemma 2.2 and integrating by parts (the eigenfunction $v$ is regular enough to justify that), we obtain

$$\beta_{s\lambda} = \int_{\Omega_0} v(\zeta \cdot \nabla \Delta v) - \int_{\Omega_0} (\Delta v)(\zeta \cdot \nabla v) + \int_{\partial\Omega_0} \frac{\partial v}{\partial \mathfrak{n}} \zeta \cdot \nabla v.$$

Using $\Delta_{\Omega_0} v + \mu_2 v = 0$ and

$$\zeta \cdot \nabla v = \frac{\partial v}{\partial \mathfrak{n}} \zeta \cdot \mathfrak{n}$$

(the latter follows from Dirichlet boundary condition), we finally obtain

$$\beta_{s\lambda}=\int_{\partial\Omega_0}\left(\frac{\partial v}{\partial \mathfrak{n}}\right)^2 \zeta\cdot\mathfrak{n}. \tag{2.11}$$

Next we look for nontrivial solutions of $\beta(s,\lambda)=0$. Noting that $\beta(0,\lambda)\equiv 0$, we define $\gamma$ by $s\gamma(s,\lambda)\equiv\beta(s,\lambda)$. Clearly $\gamma$ is of class $C^1$ and

$$\gamma(0,0)=0, \qquad \gamma_s(0,0)=\tfrac{1}{2}\beta_{ss}(0,0), \quad \gamma_\lambda(0,0)=\beta_{s\lambda}(0,0). \tag{2.12}$$

By (2.9), (2.4) and (S5), $\gamma_s(0,0)\neq 0$. Hence all solutions of $\gamma(s,\lambda)=0$ near 0 lie on a $C^1$ curve $(s(\lambda),\lambda)$. This, combined with (2.7), implies that near $(0,0)$ the set $F^{-1}(0)$ consists of the trivial branch $\{(0,\lambda)\}$ and of the curve $\{(\psi(\lambda),\lambda)\}$, where

$$\psi(\lambda)=s(\lambda)\,v+\xi(s(\lambda),\lambda).$$

We obviously have $\psi(0)=0$ and $\psi'(0)=s'(0)\,v$. We claim that $s'(0)\neq 0$, thus $\psi(\lambda)$ is a nontrivial solution for $\lambda\neq 0$ near 0. Indeed, by (2.9), (2.11),

$$s'(0)=-\frac{\gamma_\lambda(0,0)}{\gamma_s(0,0)}=-2\,\frac{\beta_{s\lambda}(0,0)}{\beta_{ss}(0,0)}=-2\,\frac{\int_{\partial\Omega_0}(\partial v/\partial\mathfrak{n})^2\,\zeta\cdot\mathfrak{n}}{f_0''(0)\int_{\Omega_0}v^3}, \tag{2.13}$$

which is different from 0 by (S4).

We now calculate the Morse indices of 0 and $\psi(\lambda)$, first in $X_e^{1/2}$ then in the full space $X^{1/2}$ (the calculation will also show that the equilibria are hyperbolic). We need to find the number of positive eigenvalues of the operators

$$(L_\lambda+\mu_2)|_{X_e}, \qquad (L_\lambda+f_0'(\psi(\lambda)))|_{X_e}. \tag{2.14}$$

For $\lambda=0$, these operators coincide with $(\Delta_{\Omega_0}+\mu_2)|_{X_e}$ and 0 is a simple eigenvalue of the latter. In fact, it is the second eigenvalue. The reason for this is that $\mu_2$ is the second eigenvalue of $-\Delta_{\Omega_0}$ (in $X$) and the first eigenvalue of $\Delta_{\Omega_0}$, $\mu_1$, corresponds to an even eigenfunction (indeed, otherwise the odd part of the eigenfunction is also an eigenfunction and it changes sign, in contradiction to the well-known property of the first eigenfunction). Thus $(\Delta_{\Omega_0}+\mu_2)|_{X_e}$ has two simple eigenvalues $\mu=0$ and $\mu=\mu_2-\mu_1>0$ and all the other eigenvalues are negative.

By standard perturbation results (see [Ka]), for $\lambda\approx 0$, each of the operators (2.14) has a unique eigenvalue (counting multiplicity) near zero, a unique eigenvalue near $\mu_2-\mu_1$ and the rest of the eigenvalues are negative. The eigenvalue near zero depends smoothly (in the $C^1$ sense) on $\lambda$ and so does the corresponding eigenfunction. More specifically, if $\mu(\lambda)$ is the eigenvalue, then one can choose the corresponding eigenfunction $v(\lambda)$ normalized in $L^2(\Omega_0)$ such that the map $\lambda \to v(\lambda)$ with values in $X_e^1$ is $C^1$.

Thus the Morse index is determined by the sign of the eigenvalue $\mu(\lambda)$. To find it, we calculate the derivative $\mu'(0)$. This is found by differentiating the equations

$$L_\lambda v(\lambda) + (\mu_2 + f_0'(z))\, v(\lambda) = \mu(\lambda)\, v(\lambda)$$

for $z = 0$, respectively for $z = \psi(\lambda)$. Using prime to denote the derivatives with respect to $\lambda$ at $\lambda = 0$, we obtain that for $z = 0$,

$$L_0 v' + (\mu_2 + f_0'(0))\, v' + \left(\frac{d}{d\lambda} L_\lambda\right)\Big|_{\lambda=0} v = \mu' v.$$

Multiplying by $v$ and integrating by parts, we obtain

$$\mu'(0) = \int_{\Omega_0} v \frac{d}{d\lambda} (L_\lambda v)\Big|_{\lambda=0}.$$

By (2.10), (2.11),

$$\mu'(0) = \beta_{\lambda s} = \int_{\Omega_0} \left(\frac{\partial v}{\partial \mathfrak{n}}\right)^2 \zeta \cdot \mathfrak{n}.$$

By (S4), $\mu'(0) > 0$, which yields

$$m_e(0) = 2 \qquad \text{for} \quad \lambda > 0, \quad \lambda \approx 0. \tag{2.15}$$

A similar calculation starting from

$$L_\lambda v(\lambda) + (\mu_2 + f_0'(\psi(\lambda))\, v(\lambda) = \mu(\lambda)\, v(\lambda)$$

results in

$$\begin{aligned} \mu'(0) &= \int_{\partial\Omega_0} \left(\frac{\partial v}{\partial \mathfrak{n}}\right)^2 \zeta \cdot \mathfrak{n} + f_0''(0) \int_{\Omega_0} \psi'(0)\, v^2 \\ &= \int_{\partial\Omega_0} \left(\frac{\partial v}{\partial \mathfrak{n}}\right)^2 \zeta \cdot \mathfrak{n} + f_0''(0)\, s'(0) \int_{\Omega_0} v^3. \end{aligned} \tag{2.16}$$

Substituting from (2.13), we obtain

$$\mu'(0) = -\int_{\partial\Omega_0} \left(\frac{\partial v}{\partial \mathfrak{n}}\right)^2 \zeta \cdot \mathfrak{n} < 0.$$

Thus

$$m_e(\psi(\lambda))=1 \qquad \text{for} \quad \lambda>0, \quad \lambda\approx 0. \tag{2.17}$$

Next we compute the Morse indices of 0 and $\psi(\lambda)$ in the full space. For that we need to calculate the derivatives of the first eigenvalues of the operators

$$(L_\lambda+\mu_2)|_{X_o}, \qquad (L_\lambda+f_0'(\psi(\lambda)))|_{X_0}. \tag{2.18}$$

These are found by differentiating the equation

$$L_\lambda w(\lambda)+(\mu_2+f_0'(z))\, w(\lambda)=\eta(\lambda)\, w(\lambda),$$

where $z=0$ or $z=\psi(\lambda)$ and $\eta(\lambda)$, $w(\lambda)$ is the first eigenvalue and the corresponding $L^2$-normalized eigenfunction, respectively. Similarly as above, we find that for $z=0$

$$\eta'(0)=\int_{\Omega_0}\left(\frac{\partial w}{\partial \mathfrak{n}}\right)^2 \zeta\cdot\mathfrak{n}.$$

By (S4), $\eta'(0)<0$ which in conjunction with (2.15) yields

$$m(0)=2.$$

For $z=\psi(\lambda)$, proceeding as in (2.16) we obtain

$$\begin{aligned}\eta'(0)&=\int_{\partial\Omega_0}\left(\frac{\partial w}{\partial \mathfrak{n}}\right)^2 \zeta\cdot\mathfrak{n}+f_0''(0)\int_{\Omega_0} w\psi'(0)\, w\\ &=\int_{\partial\Omega_0}\left(\frac{\partial w}{\partial \mathfrak{n}}\right)^2 \zeta\cdot\mathfrak{n}+f_0''(0)\, s'(0)\int_{\Omega_0} vw^2\\ &=\int_{\partial\Omega_0}\left(\frac{\partial w}{\partial \mathfrak{n}}\right)^2 \zeta\cdot\mathfrak{n}-2\,\frac{\int_{\Omega_0} vw^2}{\int_{\Omega_0} v^3}\int_{\partial\Omega_0}\left(\frac{\partial v}{\partial \mathfrak{n}}\right)^2 \zeta\cdot\mathfrak{n}\end{aligned}$$

By (S5), $\eta'(0)>0$, which together with (2.17) implies

$$m(\psi(\lambda))=2 \qquad \text{for} \quad \lambda>0, \quad \lambda\approx 0.$$

So far we have proved that, setting $\varphi=0$, $\psi=\psi(\lambda)$, statements (i), (ii), (iv) hold true for $f=f_0$ and any $\lambda>0$ sufficiently small, $\lambda<\lambda_1$, say. Note that shrinking the neighborhood $U$ has no effect on the validity of the statements, only $\lambda_1$ may have to be taken smaller. We may thus without loss of generality assume that

$$U\subset\{u\in X_e^{1/2}: \sup_x |u(x)|\leqslant 1\}$$

and that (2.1) with $f=f_0$ has no other equilibria in $\bar{U}$, the $X^{1/2}$-norm closure of $U$.

We now claim that for any $\lambda \in (0, \lambda_1)$, there is a positive constant $\delta_1(\lambda)$ such that (i), (ii), (iv) hold for any $f$ with

$$\|f - f_0\|_{C^1[-1,1]} < \delta_1(\lambda).$$

Indeed, since 0 and $\psi$ are hyperbolic, using the implicit function theorem we find two hyperbolic equilibria $\tilde{\varphi}(\lambda, f)$, $\tilde{\psi}(\lambda, f)$ of (2.1), (2.2) such that

$$\tilde{\varphi}(\lambda, f) \to 0, \qquad \tilde{\psi}(\lambda, f) \to \psi \quad \text{as} \quad \|f - f_0\|_{C^1[-1,1]} \to 0 \tag{2.19}$$

(the convergence of the equilibria is in $X_e^1$). A simple compactness argument shows that these are the only equilibria in $U$ if $f$ is sufficiently close to $f_0$. Thus (i) is satisfied. Next, the functions $f'(\tilde{\varphi}(\lambda, f)(\cdot))$ and $f'(\tilde{\psi}(\lambda, f)(\cdot))$ depend continuously in the $C(\bar{\Omega}_0)$ topology on $f \in C^1[-1, 1]$. It follows from the continuous dependence of the eigenvalues of $\Delta + f'(\tilde{\varphi}(\lambda, f))$ and $\Delta + f'(\tilde{\psi}(\lambda, f))$ on $f$ (see [16]) that the Morse indices of $\tilde{\varphi}(\lambda, f)$, $\tilde{\psi}(\lambda, f)$ are the same as the Morse indices of 0 and $\psi(\lambda)$ if $f$ is close to $f_0$. Thus (ii) and (iv) are also satisfied, which proves our claim.

We next prove that the two equilibria are connected by a heteroclinic orbit. This is a consequence of the bifurcation of equilibria, as discussed above, and is proved by an application of a center manifold theorem. The application is rather standard, however, a little caution is needed as the perturbation of the domain is involved. We give the details.

Below, $\mathrm{Lip}_{[-b,b]}$ stand for the minimal Lipschitz constant of a given function in the interval $[-b, b]$.

We make $\varepsilon > 0$ smaller, if necessary, so that the following condition is satisfied for any $\lambda \in [-\varepsilon, \varepsilon]$, $\gamma$ being a constant independent of $\lambda$. The operator $(L_\lambda + \mu_2)|_{X_e}$ has a simple eigenvalue in $(-\gamma, \gamma)$ and the rest of its spectrum is contained in $(-\infty, \gamma) \cup (\gamma, \infty)$. Let $P_\lambda$ denote the (real) spectral projection corresponding to this decomposition of the spectrum. The range of $P_\lambda$ is one-dimensional and, as $\lambda \to 0$, one has

$$P_\lambda \to P \qquad \text{in } \mathscr{L}(X_e). \tag{2.20}$$

Now, for a $C^1$ function $f$ and a constant $b > 0$, let $f_b$ be defined by

$$f_b(u) = \begin{cases} f(u) - \mu_2 u, & \text{on } [-b, b] \\ f(-b) + \mu_2 b, & \text{on } (-\infty, -b] \\ f(b) - \mu_2 b, & \text{on } [b, \infty). \end{cases} \tag{2.21}$$

Then $f_b(u)$ coincides with $f(u)-\mu_2 u$ on $[-b, b]$ and is globally Lipschitz with

$$\operatorname{Lip} f_b = \operatorname{Lip}_{[-b,b]}(f-\mu_2).$$

Consider the modified equation

$$u_t = (L_\lambda + \mu_2)\, u + f_b(u), \qquad u(t) \in X_e. \tag{2.22}$$

Due to the above spectral properties of $(L_\lambda + \mu_2)|_{X_e}$, we can apply a center manifold theorem to (2.22). More specifically, the following assertion holds true:

LEMMA 2.3. *There is a $\delta_2 > 0$ such that for any $\lambda \in [-\varepsilon, \varepsilon]$ and any $f$ satisfying*

$$\operatorname{Lip}_{[-b,b]}(f-\mu_2) < \delta_2 \tag{2.23}$$

*for some $b > 0$, there is a Lipschitz function*

$$\sigma_{\lambda, f_b}\colon R(P_\lambda) \to X_e^{1/2} \cap (R(I-P_\lambda))$$

*with the following properties*:

(M1) *The manifold*

$$W_{\lambda, f_b} := \{\xi + \sigma_{\lambda, f_b}(\xi) : \xi \in R(P_\lambda)\} \tag{2.24}$$

*is invariant under the semiflow of* (2.22). *More specifically, for any $z_0 \in W_{\lambda, f_b}$ there is a solution $u(t)$ of* (2.22) *defined for all $t \in \mathbb{R}$ such that $u(0) = z_0$ and $u(t) \in W_{\lambda, f_b}$.*

(M2) *$W_{\lambda, f_b}$ contains all trajectories of* (2.22) *that are defined for all $t \in \mathbb{R}$ and bounded in $X_e^{1/2}$. In particular, it contains all equilibria of* (2.22).

(M3) *One has* $\operatorname{Lip} \sigma_{\lambda, f_b} < c$ *for a constant $c$ independent of $\lambda \in [-\varepsilon, \varepsilon]$ and $f$ satisfying* (2.23).

*Proof.* A Lipschitz graph $W_{\lambda, f_b}$ with properties (M1), (M2) is constructed using the Lyapunov–Perron integral operator as in [Va2; Va-I; Ry] (see also [Cho-L; Cho-L-L; Che-C-H; Che-H-T] for a similar construction for center unstable and other manifolds). One needs $\operatorname{Lip} f_b$ to be sufficiently small for the integral operator to define a contraction on an appropriate space. How small it actually has to be depends on the operator $L_\lambda$. However, it can be seen easily, by inspecting the specific conditions on $f_b$ in the above references and using perturbation results [16], that the contraction property is unaffected by a perturbation of the operator which is sufficiently small in the norm $\mathscr{L}(X_e^1, X_e)$. Thus the continuity of $\lambda \to L_\lambda$

guarantees that we can take $\delta_2 > 0$ independently of $\lambda \in [-\varepsilon, \varepsilon]$. This also guarantees that (M3) is satisfied, as the Lipschitz constant of $\sigma_{\lambda, f_b}$ is estimated in terms of $c$. ∎

We now complete the proof of Theorem 2.1. We shall rely on the following simple observation. Equip the one-dimensional graph $W_{\lambda, f_b}$ with an order structure isomorphic to the natural ordering of $\mathbb{R}$. Since the Lipschitz constant of the function $\sigma_{\lambda, f_\varepsilon}$ is bounded by a constant $c$ independent of $f$ and $\lambda$, and $P_\lambda$ depends continuously on $\lambda$, there is a neighborhood $U_0 \subset U$ of 0 with the following property. If $e_1, e_2$ are two points in $W_{\lambda, f_b} \cap U_0$ then the order interval $[e_1, e_2]$ in $W_{\lambda, f_b}$ is contained in $U$.

Now, if we take $\lambda \in (0, \varepsilon)$ sufficiently small, then the equilibrium $\psi(\lambda) = \tilde{\psi}(\lambda, f_0)$ is contained in $U_0$. Fix any such $\lambda$. Choose $b \in (0, 1)$ so small that $f = f_0$ satisfies

$$\operatorname{Lip} \sigma_{\lambda, f_b} < \delta_2 \tag{2.25}$$

with $\delta_2$ as in Lemma 2.3 (recall that $f'_0(0) - \mu_2 = 0$). Let $\delta(\lambda)$ be so small that the following requirements are satisfied:

- $\delta(\lambda) < \max\{\delta_1(\lambda), \delta_2\}$,
- for any $f$ with

$$\| f - f_0 \|_{C^1[-1, 1]} < \delta(\lambda)$$

(2.25) is satisfied,

- the equilibria $\tilde{\varphi}(\lambda, f)$, $\tilde{\psi}(\lambda, f)$ are contained in $U_0$ (this choice is possible by (2.19)).

By (M2), both the equilibria $\tilde{\varphi}(\lambda, f)$, $\tilde{\psi}(\lambda, f)$ are contained in the center manifold $W_{\lambda, f_b}$. By the above observation, the segment of the manifold between the equilibria is contained in $U$. Now, as $f_b$ coincides with $f$ on $U$ and (2.1) has no other equilibria in $U$, the segment in the invariant manifold must actually be a heteroclinic orbit of (2.1) joining the two equilibria.

We have thus established existence of a heteroclinic orbit, as asserted in (iii) of Theorem 2.1. The proof is now complete. ∎

## 3. DOMAINS SATISFYING THE HYPOTHESES

In this section we find a domain $\Omega_0 \subset \mathbb{R}^2$ and a function $\zeta$ for which the hypotheses (S1), (S2)′, (S3)–(S5) are satisfied.

A natural choice of a symmetric domain that satisfies (S3) (the second eigenvalue of the Laplacian is a double eigenvalue with both even an odd eigenfunctions) is the disk

$$B=\{x\in\mathbb{R}^2\colon |x|<1\}.$$

However, $B$ is "too symmetric" and with $\Omega_0=B$ one cannot meet the hypotheses (S5). We therefore perturb the disk, removing the symmetry in $x_1$ while keeping the symmetry in $x_2$ and preserving the multiplicity of the second eigenvalue.

We consider domains of the form $h(B)$ where $h$ is a near identity diffeomorphism of $\mathbb{R}^2$. For a positive integer $k$ let $\mathscr{Y}^k$ denote the space of all functions $g\colon \mathbb{R}^2\to\mathbb{R}^2$ with all derivatives up to order $k$ bounded equipped with the norm

$$\|g\|_{\mathscr{Y}^k}:=\sup_{x\in\mathbb{R}^2,\, i=1,\dots,k}|D^ig(x)|.$$

THEOREM 3.1. *For any integer $k\geqslant 3$ and any $\varepsilon>0$ there is a smooth diffeomorphism $h=(h_1,h_2)\colon\mathbb{R}^2\to\mathbb{R}^2$ such that*

$$h_1(x_1,-x_2)=h_1(x_1,x_2),\qquad h_2(x_1,-x_2)=-h_2(x_1,x_2), \tag{3.1}$$

$$\|h-I\|_{\mathscr{Y}^k}<\varepsilon\qquad (i=0,1,\dots,k), \tag{3.2}$$

*and* (S1), (S2)′, (S3)–(S5) *are satisfied with $\Omega_0=h(B)$ and some smooth function $\zeta\colon\mathbb{R}^2\to\mathbb{R}^2$.*

Note that if $\varepsilon$ is sufficiently small then the domain $\Omega_0$ is convex.

The proof of the theorem is carried out in several lemmas given after a preliminary discussion.

We use notation as in Section 2,

$$X=L^2(B),\qquad X^1=H^2(B)\cap H^1_0(B);$$

$X_e$, $X_o$ are the subspaces of $X^1$ consisting of all functions that are even and odd in $x_2$, respectively, $X^1_e=X_e\cap X^1$ and $X^1_o=X_o\cap X^1$. (In this section we consider linear equations only so there is no need to work with the $L^p$ spaces.)

For a diffeomorphism $h$ satisfying (3.1) let $\Delta_{h(B)}$ denote the $L^2$ realization of the Laplacian on $h(B)$ under Dirichlet boundary conditions:

$$D(\Delta_{h(B)})=H^2(h(B))\cap H^1_0(h(B)),$$

$$(\Delta_{h(B)}u)(x)=\Delta u(x)\qquad (x\in h(B)).$$

Let

$$L(h) = L_h = h^* \Delta_{h(B)} (h^*)^{-1}.$$

Note that

$$h \mapsto L(h) \colon \mathscr{Y}^k \to \mathscr{L}(X^1, X)$$

is a smooth map.

Let $\mu(h)$ denote the second eigenvalue of $-L_h|_{X_e}$ and $\nu(h)$ the first eigenvalue of $-L_h|_{X_o}$. For $h = I$, one has $\mu(I) = \nu(I) = \mu_2$, the second eigenvalue of the Laplacian. The eigenspace of $\mu_2$ is spanned by the eigenfunctions $v_1$, $w_1$ which are explicitly given by

$$\begin{aligned} v_1(r\cos\theta, r\sin\theta) &= J(r)\cos\theta, \\ w_1(r\cos\theta, r\sin\theta) &= J(r)\sin\theta \qquad (r \geqslant 0, \theta \in [-\pi, \pi)). \end{aligned} \tag{3.3}$$

Here $J(r)$ is a Bessel function satisfying $J(r) > 0$ $(r \in (0, 1))$ and $J(1) = 0$. We normalize $J$ such that $v_1$, $w_1$ have the $L^2(B)$ norm equal to 1. Now, $\mu_2$ is a simple eigenvalue for the restriction of the Laplacian to any of the spaces $X_e$, $X_o$. By continuity properties of the spectrum [Ka], for $h$ sufficiently close to the identity, $\mu(h)$, $\nu(h)$ are close to $\mu_2$ and remain simple. We denote by $v(h)$, $w(h)$ the eigenfunctions corresponding to $\mu(h)$, $\nu(h)$, respectively, normalized in the $L^2(B)$ norm and such that $v(h) > 0$ for $x_2 > 0$ and $w(h) > 0$ for $x_1 > 0$. This defines the eigenfunctions uniquely for $h$ near $I$. Notice that the functions $\tilde{v} = (h^*)^{-1} v(h)$, $\tilde{w} = (h^*)^{-1} w(h)$ are eigenfunctions of $\Delta_{h(B)}$ corresponding to the eigenvalues $\mu(h)$, $\nu(h)$. Due to simplicity of the eigenvalues, perturbation theorems of [Ka] imply that $\mu(h)$, $\nu(h)$ are $C^1$ functions of $h$. Similarly the maps

$$h \mapsto v(h) \in X_e^1, \qquad h \mapsto w(h) \in X_o^1$$

are of class $C^1$. Combining this with elliptic regularity, we also obtain the $C^1$ dependence when $v(h)$, $w(h)$ are viewed as functions with values in $C^1(\bar{B})$.

Consider the integrals

$$\begin{aligned} \mathfrak{I}(h) &:= \int_{h(B)} \tilde{v}(h)^3 = \int_B v(h)^3\, |Dh|, \\ \mathfrak{K}(h) &:= \int_{h(B)} \tilde{v}(h)\, \tilde{w}(h)^2 = \int_B v(h)\, w(h)^2\, |Dh|, \end{aligned} \tag{3.4}$$

where $|Dh|$ is the Jacobian of $h$ (it is positive for $h$ near the identity). If we want to satisfy (S4), (S5) for $\Omega_0 = h(B)$, we necessarily need

$$\frac{\mathfrak{R}(h)}{\mathfrak{I}(h)} < 0. \tag{3.5}$$

Also, in view of (S3), we require

$$\mu(h) = \nu(h).$$

We first find a smooth function $h$ satisfying these requirements. Having found such an $h$ sufficiently close to the identity, we will then be able to find a function $\zeta$ such that all the hypotheses (S1), (S2)′, (S3)–(S5) are satisfied (with $\Omega_0 = h(B)$).

Let $\mathscr{Y}^k_s$ be the closed subspace of $\mathscr{Y}^k$ consisting of all the functions that satisfy the symmetry requirement (3.1).

LEMMA 3.2. *There is a submanifold $M$ of $\mathscr{Y}^k_s$ $(k \geqslant 3)$ of codimension* 1 *such that $I \in M$ and for any $h \in M$ one has $\mu(h) = \nu(h)$.*

*If $\bar{h}$ is given by*

$$\bar{h}(r\cos\theta, r\sin\theta) = \beta(r)\, m(\theta)(\cos\theta, \sin\theta) \qquad (r \geqslant 0, \theta \in \mathbb{R}) \tag{3.6}$$

*where $m$, $\beta$ are $C^k$ functions, $m$ is even and $2\pi$-periodic, $\beta$ is identical to* 0 *near $r = 0$ and $\beta(1) = 1$, then $\bar{h}$ is contained in $T_I M$, the tangent space of $M$ at $I$, if and only if*

$$\int_{-\pi}^{\pi} m(\theta)\cos^2\theta\, d\theta - \int_{-\pi}^{\pi} m(\theta)\sin^2\theta\, d\theta = 0. \tag{3.7}$$

*Proof.* The first assertion follows from the implicit function theorem, provided we show that the derivative of the map $h \mapsto \mu(h) - \nu(h)$ at $h = I$ is a nonzero functional. The tangent space $T_I M$ then consists of all $\bar{h} \in \mathscr{Y}^k$ such that

$$\mu'(I)\,\bar{h} - \nu'(I)\,\bar{h} = 0. \tag{3.8}$$

The calculation of the derivative with $\bar{h}$ given by (3.6) is carried out in the next lemma. From the formulas given in that lemma, it is obvious that the functional is nonzero and that (3.7) is equivalent to (3.8) for such functions. ∎

LEMMA 3.3. *For $\bar{h}$ as in* (3.6) *one has*

$$\mu'(I)\,\bar{h} = -(J'(1))\int_{-\pi}^{\pi} m(\theta)\cos^2\theta\,\mathrm{d}\theta$$
$$\nu'(I)\,\bar{h} = -(J'(1))\int_{-\pi}^{\pi} m(\theta)\sin^2\theta\,\mathrm{d}\theta, \tag{3.9}$$

*where $J$ is the Bessel function as in* (3.3) *and $J'(1)\neq 0$. Furthermore, the derivatives of the eigenfunctions*

$$y = Dv(I)\,\bar{h}, \qquad z = Dv(I)\,\bar{h}$$

*are the* (*uniquely determined*) *solutions of the problems*

$$\Delta y + \mu_2 y = -[\bar{h}\cdot\nabla, \Delta]\, v_1 - (\mu'(I)\,\bar{h})\, v_1,$$
$$y \in X_e^1, \qquad \int_B y v_1 = 0, \tag{3.10}$$

$$\Delta z + \mu_2 z = -[\bar{h}\cdot\nabla, \Delta]\, w_1 - (\nu'(I)\,\bar{h})\, w_1,$$
$$z \in X_o^1, \qquad \int_B z w_1 = 0, \tag{3.11}$$

*where $[\bar{h}\cdot\nabla, \Delta] = \zeta\cdot\nabla\Delta - \Delta(\zeta\cdot\nabla)$ is the commutator of $\bar{h}\cdot\nabla$ and $\Delta$* (*as in Lemma* 2.2).

*Proof.* The calculations are similar to those carried out in Section 2 for evaluation of the Morse indices. Differentiating

$$L(h)\,v(h) + \mu(h)\,v(h) = 0, \qquad \int_B v(h)^2 = 1$$

and substituting $h = I$, we obtain

$$\Delta y + \mu_2 y = -(DL(I)\,\bar{h} + \mu'(I)\,\bar{h})\, v_1,$$
$$\int_B y v_1 = 0. \tag{3.12}$$

Multiplying by $v_1$ and integrating, we obtain

$$\mu'(I)\,\bar{h} = -\int_B (DL(I)\,\bar{h})\, v_1.$$

Lemma 2.2 and integration by parts yield (cf. (2.11))

$$\mu'(I)\,\bar{h} = -\int_{\partial B} \left(\frac{\partial v_1}{\partial \mathfrak{n}}\right)^2 \bar{h}\cdot \mathfrak{n},$$

where $\mathfrak{n}(x, y) = (x, y)$ is the unit outward normal vector field on $\partial B$. Using polar coordinates and (3.3), (3.6) we obtain (3.9). We have $J'(1) \neq 0$ because $J(1) = 0$ and the positive roots of the Bessel function are simple. Relations (3.10) follow from (3.12) and Lemma 2.2. The calculation for $v(h)$ and $w(h)$ are similar and are omitted. ∎

We now take a $C^1$ curve

$$h_\gamma = I + \gamma \bar{h} + o(\gamma) \qquad (\gamma \approx 0)$$

in the manifold $M$. We want to choose the tangent $\bar{h}$ such that the derivatives of the functions

$$\mathfrak{j}\colon \gamma \mapsto \mathfrak{I}(h_\gamma), \qquad \mathfrak{k}\colon \gamma \mapsto \mathfrak{K}(h_\gamma)$$

at $\gamma = 0$ have opposite signs ($\mathfrak{I}$, $\mathfrak{K}$ are as in (3.3)). This will guarantee that (3.5) is satisfied for $h = h_\gamma$ arbitrarily close to the identity. Note that the functions $\mathfrak{j}$, $\mathfrak{k}$ are $C^1$ near $\gamma = 0$ because $h \mapsto v(h), w(h) \in X^1 \hookrightarrow C(\bar{B})$ are $C^1$. We calculate their derivatives in the following lemma.

LEMMA 3.4. *Let $h_\gamma$ be a $C^1$ curve as above. Let $\xi$, $\eta$, $\rho$ be the (uniquely determined) solutions of the problems*

$$\begin{aligned} &\Delta\xi + \mu_2 \xi = v_1^2, \\ &\xi \in X_e^1, \qquad \int_B \xi v_1 = 0; \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\Delta\eta + \mu_2 \eta = w_1^2, \\ &\eta \in X_e^1, \qquad \int_B \eta v_1 = 0; \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\Delta\rho + \mu_2 \rho = v_1 w_1, \\ &\rho \in X_o^1, \qquad \int_B \rho w_1 = 0. \end{aligned} \tag{3.15}$$

*Then*

$$\mathfrak{j}'(0) = -3 \int_{\partial B} \frac{\partial \xi}{\partial \mathfrak{n}} \frac{\partial v_1}{\partial \mathfrak{n}} \bar{h} \cdot \mathfrak{n} \tag{3.16}$$

$$\mathfrak{k}'(0) = -2 \int_{\partial B} \frac{\partial \rho}{\partial \mathfrak{n}} \frac{\partial w_1}{\partial \mathfrak{n}} \bar{h} \cdot \mathfrak{n} - \int_{\partial B} \frac{\partial \eta}{\partial \mathfrak{n}} \frac{\partial v_1}{\partial \mathfrak{n}} \bar{h} \cdot \mathfrak{n}. \tag{3.17}$$

*For* $\bar{h}$ *given by* (3.6), *these formulas yield*

$$\mathfrak{j}'(0) = -3J_r(1) \int_{-\pi}^{\pi} \xi_r(1, \theta)\, m(\theta) \cos\theta \,\mathrm{d}\theta, \tag{3.18}$$

$$\mathfrak{k}'(0) = -J_r(1) \int_{-\pi}^{\pi} (2\rho_r(1, \theta) \sin\theta + \eta_r(1, \theta) \cos\theta)\, m(\theta) \,\mathrm{d}\theta. \tag{3.19}$$

*Proof.* First observe that in each of problems (3.13)–(3.15), the right-hand side is $L^2(B)$ orthogonal to the kernel of the left-hand side. Thus, by the Fredholm alternative, the solution exists, and the given conditions determine it uniquely.

Now

$$\mathfrak{j}'(0) = \mathfrak{J}'(I)\, \bar{h} = 3 \int_B v_1^2\, Dv(I)\, \bar{h} + \int_B v_1^3 \operatorname{div} \bar{h}.$$

With $y = Dv(I)\, \bar{h}$ and $\xi$ as in (3.13), we have (making use of integration by parts, Lemma 3.3, and the equalities $\Delta v_1 = -\mu_2 v_1$, $\int \xi v_1 = 0$)

$$\begin{aligned}
\int_B v_1^2 y &= \int_B ((\Delta + \mu_2)\, \xi)\, y = \int_B \xi (\Delta + \mu_2)\, y \\
&= \int_B \xi(\Delta(\bar{h} \cdot \nabla v_1) - \bar{h} \cdot \nabla \Delta v_1) = \int_B \xi(\Delta + \mu_2)(\bar{h} \cdot \nabla v_1) \\
&= \int_B ((\Delta + \mu_2)\, \xi)\, \bar{h} \cdot \nabla v_1 - \int_{\partial B} \frac{\partial \xi}{\partial \mathfrak{n}} \bar{h} \cdot \nabla v_1 \\
&= \int_B v_1^2 \bar{h} \cdot \nabla v_1 - \int_{\partial B} \frac{\partial \xi}{\partial \mathfrak{n}} \frac{\partial v_1}{\partial \mathfrak{n}} \bar{h} \cdot \mathfrak{n}.
\end{aligned}$$

Thus

$$\mathfrak{j}'(0) = \int_B \operatorname{div}(h v_1^3) - 3 \int_{\partial B} \frac{\partial \xi}{\partial \mathfrak{n}} \frac{\partial v_1}{\partial \mathfrak{n}} \bar{h} \cdot \mathfrak{n} = -3 \int_{\partial B} \frac{\partial \xi}{\partial \mathfrak{n}} \frac{\partial . v_1}{\partial \mathfrak{n}} \bar{h} \cdot \mathfrak{n}.$$

For the derivative of $\mathfrak{k}$, we have

$$\mathfrak{k}'(0) = 2\int_B w_1 v_1\, Dw(I)\,\bar{h} + \int_B w_1^2\, Dv(I)\,\bar{h} + \int_B w_1^2 v_1 \operatorname{div} \bar{h}.$$

Let $y = Dv(I)\,\bar{h}$, as above, $z = Dw(I)\,\bar{h}$ and let $\eta$, $\rho$ be as in (3.14), (3.15). Calculations similar to those above yield

$$\begin{aligned}
&2\int_B ((\Delta+\mu_2)\,\rho)\,z + \int_B ((\Delta+\mu)\,\eta)\,y \\
&\quad = 2\int_B \rho(\Delta+\mu_2)\,z + \int_B \eta(\Delta+\mu)\,y \\
&\quad = 2\int_B \rho(\Delta(\bar{h}\cdot\nabla w_1) - \bar{h}\cdot\nabla\Delta w_1) + \int_B \eta((\Delta\bar{h}\cdot\nabla v_1) - \bar{h}\cdot\nabla\Delta v_1) \\
&\quad = 2\int_B \rho(\Delta+\mu_2)(\bar{h}\cdot\nabla w_1) + \int_B \eta(\Delta+\mu_2)(\bar{h}\cdot\nabla v_1) \\
&\quad = 2\left(\int_B ((\Delta+\mu_2)\,\rho)\,h\cdot\nabla w_1 - \int_{\partial B} \frac{\partial\rho}{\partial\mathfrak{n}}\,\frac{\partial w_1}{\partial\mathfrak{n}}\,\bar{h}\cdot\mathfrak{n}\right) \\
&\qquad + \int_B ((\Delta+\mu_2)\,\eta)\,h\cdot\nabla v_1 - \int_{\partial B} \frac{\partial\eta}{\partial\mathfrak{n}}\,\frac{\partial v_1}{\partial\mathfrak{n}}\,\bar{h}\cdot\mathfrak{n} \\
&\quad = 2\int_B v_1 w_1 \bar{h}\cdot\nabla w_1 + \int_B w_1^2 \bar{h}\cdot\nabla v_1 \\
&\qquad - 2\int_{\partial B} \frac{\partial\rho}{\partial\mathfrak{n}}\,\frac{\partial w_1}{\partial\mathfrak{n}}\,\bar{h}\cdot\mathfrak{n} - \int_{\partial B} \frac{\partial\eta}{\partial\mathfrak{n}}\,\frac{\partial v_1}{\partial\mathfrak{n}}\,\bar{h}\cdot\mathfrak{n}
\end{aligned}$$

Hence

$$\mathfrak{k}'(0) = \int_B \operatorname{div}(\bar{h} w_1^2 v_1) - 2\int_{\partial B} \frac{\partial\rho}{\partial\mathfrak{n}}\,\frac{\partial w_1}{\partial\mathfrak{n}}\,\bar{h}\cdot\mathfrak{n} - \int_{\partial B} \frac{\partial\eta}{\partial\mathfrak{n}}\,\frac{\partial v_1}{\partial\mathfrak{n}}\,\bar{h}\cdot\mathfrak{n}$$

and (3.17) follows. Using polar coordinates in (3.16), (3.17) and substituting from (3.3), (3.6), we obtain (3.18), (3.19). ∎

LEMMA 3.5. *Let $\xi$, $\eta$, $\rho$ be as in Lemma 3.4. Then, in polar coordinates,*

$$\begin{aligned}
\xi(r,\theta) &= a(r) + b(r)\cos 2\theta, \\
\eta(r,\theta) &= a(r) - b(r)\cos 2\theta, \qquad (3.20) \\
\rho(r,\theta) &= b(r)\sin 2\theta,
\end{aligned}$$

*for some $C^1$ functions $a$, $b$, with $b'(1) < 0$.*

*Proof.* We have

$$v_1^2 = J^2(r)\cos^2\theta = J^2(r)\,\frac{1+\cos 2\theta}{2},$$

$$w_1^2 = J^2(r)\sin^2\theta = J^2(r)\,\frac{1-\cos 2\theta}{2},$$

$$v_1 w_1 = J^2(r)\,\frac{\sin 2\theta}{2}.$$

Three subspaces of $L^2(B)$ consisting of all functions of the form $a(r)$, $b(r)\cos 2\theta$, $c(r)\sin 2\theta$, respectively, are invariant under $\Delta_B$. Moreover, $-\mu_2$ is not an eigenvalue of the restriction of $\Delta_B$ to any of these invariant spaces ($-\mu_2$ is a double eigenvalue with the eigenspace spanned by $v_1 = J(r)\cos\theta$, $w_1 = J(r)\sin\theta$). These observations and Eqs. (3.13)–(3.15) imply that the first two relations in (3.20) hold for some functions $a(r)$, $b(r)$ and that

$$\rho(r,\theta) = c(r)\sin 2\theta$$

for some function $c(r)$. We also obtain that the function $u := b(r)\cos 2\theta$ is a solution of

$$\begin{aligned} \Delta u + \mu_2 u &= p, & x &\in \tilde{B}, \\ u &= 0, & x &\in \partial\tilde{B}, \end{aligned} \tag{3.21}$$

where

$$\tilde{B} = \left\{ (r\cos\theta, r\sin\theta) : r\in[0,1),\ -\frac{\pi}{4} < \theta < \frac{\pi}{4} \right\}$$

and

$$p(r\cos\theta, r\sin\theta) = J^2(r)\,\frac{\cos 2\theta}{2}.$$

Note that $p$ is positive in $\tilde{B}$. Further, the first eigenvalue of $-\Delta$ on $\tilde{B}$ under Dirichlet boundary condition is greater than $\mu_2$ (for $\mu_2$ is the first eigenvalue on the larger domain $\{-\pi/2 < \theta < \pi/2\}$). This justifies an application of a theorem from the theory of positive operators (see [Kr, Sections 2.5, 7.2]) which says that (3.21) has a unique solution and this solution is nonnegative. Thus $u$ is nonnegative in $\tilde{B}$, obviously not identical to zero in $\tilde{B}$, and assumes its minimum value 0 at any point of $\{(r\cos\theta, r\sin\theta) : r=1, -\pi/4<\theta<\pi/4\}$. By an extension of the Hopf lemma (see [Gi-N-N]), on this set the derivative of $u$ at outward directions is negative, which implies $b'(1)<0$, as asserted.

Finally observe that the function

$$\tilde{\rho}(r, \theta) = c(r)\cos 2\theta = \rho\left(r, \theta - \frac{\pi}{4}\right)$$

is a solution of the same problem (3.21). Therefore, by uniqueness, $b(r) = c(r)$. ▮

LEMMA 3.6. *Let $\bar{h}$ be as in Lemma 3.2. Assume that the (odd $2\pi$-periodic function) $m(\theta)$ satisfies the relations*

$$\int_{-\pi}^{\pi} m(\theta)\, d\theta = \int_{-\pi}^{\pi} m(\theta)\cos\theta\, d\theta = \int_{-\pi}^{\pi} m(\theta)\cos 2\theta\, d\theta = 0 \tag{3.22}$$

*and*

$$J_r(1)\int_{-\pi}^{\pi} m(\theta)\cos 3\theta\, d\theta > 0 \tag{3.23}$$

*(for example take $m(\theta) = J_r(1)\cos 3\theta$). Then $\bar{h}\in T_I M$ and if $\bar{h}_\gamma = I+\gamma h+o(\gamma)$ ($\gamma$ near 0) is a $C^1$ curve in $M$ tangent to $\bar{h}$ then*

$$\mathfrak{i}'(0) > 0 > \mathfrak{t}'(0).$$

*Proof.* Relations (3.22) imply

$$\int_{-\pi}^{\pi} m(\theta)\cos^2\theta\, d\theta = \int_{-\pi}^{\pi} m(\theta)\sin^2\theta\, d\theta = 0.$$

Thus $\bar{h}\in T_I M$ by Lemma 3.2.

Combining the formulas from Lemmas 3.4, 3.5, we have

$$\begin{aligned}\mathfrak{i}'(0) &= -3J_r(1)\int_{-\pi}^{\pi} \xi_r(1,\theta)\, m(\theta)\cos\theta\, d\theta \\ &= -3J_r(1)\left(a'(1)\int_{-\pi}^{\pi} m(\theta)\cos\theta\, d\theta + b'(1)\int_{-\pi}^{\pi} m(\theta)\cos\theta\cos 2\theta\, d\theta\right),\end{aligned}$$

and

$$\mathfrak{k}'(0) = -J_r(1) \int_{-\pi}^{\pi} (2\rho_r(1, \theta) \sin \theta + \eta_r(1, \theta) \cos \theta)\, m(\theta)\, \mathrm{d}\theta$$

$$= -J'(1) \left( 2b'(1) \int_{-\pi}^{\pi} m(\theta) \sin 2\theta \sin \theta \,\mathrm{d}\theta + a'(1) \int_{-\pi}^{\pi} m(\theta) \cos \theta \,\mathrm{d}\theta \right.$$

$$\left. - b'(1) \int_{-\pi}^{\pi} m(\theta) \cos \theta \cos 2\theta \,\mathrm{d}\theta \right).$$

Substituting $\cos 2\theta \cos \theta = \frac{1}{2}(\cos 3\theta + \cos \theta)$ and $\sin 2\theta \sin \theta = \frac{1}{2}(\cos \theta - \cos 3\theta)$,

$$\mathfrak{j}'(0) = -3J_r(1) \left( (a'(1) + \tfrac{1}{2}b'(1)) \int_{-\pi}^{\pi} m(\theta) \cos \theta \,\mathrm{d}\theta \right.$$

$$\left. + \tfrac{1}{2}b'(1) \int_{-\pi}^{\pi} m(\theta) \cos 3\theta \,\mathrm{d}\theta \right)$$

$$\mathfrak{k}'(0) = -J_r(1) \left( (a'(1) + \tfrac{1}{2}b'(1)) \int_{-\pi}^{\pi} m(\theta) \cos \theta \,\mathrm{d}\theta \right.$$

$$\left. - \tfrac{3}{2}b'(1) \int_{-\pi}^{\pi} m(\theta) \cos 3\theta \,\mathrm{d}\theta \right).$$

Relations (3.22), (3.23), together with $b'(1) < 0$ now give

$$\mathfrak{j}'(0) > 0 > \mathfrak{k}'(0). \quad \blacksquare$$

*Completion of the proof of Theorem* 3.1. Let $\bar{h}$ be as in (3.6) with both functions $\beta$ and $m$ smooth (that is, of class $C^\infty$), $\beta(1) = 1$, $\beta \equiv 0$ for $r$ near 0, $m$ even, $2\pi$-periodic and such that the relations (3.22), (3.23) are satisfied. By the last lemma $\bar{h} \in T_I M$, so there is a $C^1$ curve

$$h_\gamma = I + \gamma \bar{h} + o(\gamma) \qquad (\gamma \approx 0)$$

in $M$. Moreover, as $\bar{h}$ is smooth, we can find such a curve with the additional property that $h_\gamma$ is a smooth function for any $\gamma$. Indeed, take any two dimensional subspace $\operatorname{span}\{\bar{h}, \bar{g}\}$ of $\mathscr{Y}^k$, spanned by $\bar{h}$ and another smooth function $\bar{g}$ not contained in $T_I M$. The intersection of this space with (the codimension-one manifold) $M$ contains a curve with the desired smoothness property.

We show that, with an appropriate choice of $\zeta$, hypotheses (S1), (S2)$'$, (S3)–(S5) are all satisfied for $\Omega_0 = h_\gamma(B)$ with $|\gamma|$ nonzero and sufficiently small.

First note that as $h_\gamma \in M$, hypotheses (S1), (S3) are satisfied (cf. Lemma 3.2).

Next we define a smooth function $\zeta\colon \mathbb{R}^2 \to \mathbb{R}^2$. Set

$$\zeta((r\cos\theta, r\sin\theta) = q(r)\, k(\theta)(\cos\theta, \sin\theta) \qquad (r \geqslant 0, \theta \in \mathbb{R}), \tag{3.24}$$

where $q$, $k$ are smooth $q(1) = 1$, $q \equiv 0$ near $r = 0$ and $k$ is $2\pi$-periodic and even. Note that $\zeta$ satisfies the symmetry hypothesis (S2)$'$. We further require that $k(\theta)$ satisfies

$$\begin{aligned} &\int_{-\pi}^{\pi} \cos^2\theta k(\theta)\, d\theta = c > 0, \\ &\int_{-\pi}^{\pi} \sin^2\theta k(\theta)\, d\theta = -1, \end{aligned} \tag{3.25}$$

where $c$ is a sufficiently large constant (as specified below). It is possible to find such $k(\theta)$ since $\sin^2\theta$ and $\cos^2\theta$ are linearly independent even functions. In fact, we can assign arbitrary values to the integrals by adjusting $k(\theta)$ (one can for example take $k(\theta)$ to be an appropriate linear combination of $\sin^2\theta$ and $\cos^2\theta$).

Let us now consider the integrals that appear in hypothesis (S4):

$$\begin{aligned} d_1(\gamma) &:= \int_{\partial(h_\gamma(B))} \left(\frac{\partial \tilde{v}(h_\gamma)}{\partial \mathfrak{n}}\right)^2 \zeta \cdot \mathfrak{n}, \\ d_2(\gamma) &:= \int_{\partial(h_\gamma(B))} \left(\frac{\partial \tilde{w}(h_\gamma)}{\partial \mathfrak{n}}\right)^2 \zeta \cdot \mathfrak{n}. \end{aligned} \tag{3.26}$$

Recall that $\tilde{v}(h) = (h^*)^{-1} v(h)$, $\tilde{w}(h) = (h^*)^{-1} w(h)$ are eigenfunctions of $\Delta_{h(B)}$ corresponding to the eigenvalues $\mu(h)$, $\nu(h)$. Viewed as functions with values in $C^1(\bar{B})$, $v(h)$, $w(h)$ are of class $C^1$ with respect to $h$. It is therefore easy to see that, as $\gamma \to 0$,

$$d_1(\gamma) \to d_1(0) = \int_{\partial B} \left(\frac{\partial v_1}{\partial \nu}\right)^2 \zeta \cdot \mathfrak{n},$$

$$d_2(\gamma) \to d_2(0) = \int_{\partial B} \left(\frac{\partial w_1}{\partial \mathfrak{n}}\right)^2 \zeta \cdot \mathfrak{n}.$$

Using polar coordinates and substituting from (3.3), (3.24), we obtain

$$d_1(0) = (J'(1))^2 \int_{-\pi}^{\pi} k(\theta) \cos^2 \theta \, \mathrm{d}\theta,$$
$$d_2(0) = (J'(1))^2 \int_{-\pi}^{\pi} k(\theta) \sin^2 \theta \, \mathrm{d}\theta.$$

By (3.25), the above convergence property implies

$$d_1(\gamma) > 0 > d_2(\gamma),$$

for $\gamma \approx 0$, so that (S4) is satisfied.

Finally, consider the value

$$d_2(\gamma) - 2 \frac{\mathfrak{k}(\gamma)}{\mathfrak{j}(\gamma)} d_1(\gamma)$$

that appears in hypothesis (S5). As $\gamma \to 0$, this value converges to

$$d_2(0) - 2 \frac{\mathfrak{k}'(0)}{\mathfrak{j}'(0)} d_1(0). \tag{3.27}$$

By Lemma 3.6,

$$\frac{\mathfrak{k}'(0)}{\mathfrak{j}'(0)} < 0.$$

Thus, choosing $c$ in (3.27) sufficiently large, we make the limit value (3.27) positive. This implies that (S5) is satisfied for $|\gamma| \neq 0$ sufficiently small. The proof is complete. ∎

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## REFERENCES

[Am] H. Amann, "Linear and Quasilinear Parabolic Problems," Birkhäuser, Berlin, 1995.

[An] S. B. Angenent, The Morse–Smale property for a semilinear parabolic equation, *J. Differential Equations* **62** (1986), 427–442.

[Br-P] P. Brunovský and P. Poláčik, Morse Smale structure of generic reaction diffusion equations in higher space dimension, *J. Differential Equations* **135** (1997), 129–181.

[Che-C-H] M. Chen, X.-Y. Chen, and J. K. Hale, Structural stability for time-periodic one-dimensional parabolic equations, *J. Differential Equations* **96** (1992), 355–418.

[Che-H-T] X.-Y. Chen, J. K. Hale, and B. Tan, Foliations of $C^1$ semigroups in Banach spaces, *J. Differential Equations* **139** (1997), 283–318.

[Cho-H] S.-N. Chow and J. K. Hale, "Methods of Bifurcation Theory," Springer, New York, 1982.

[Cho-L] S.-N. Chow and K. Lu, Invariant manifolds for flows in Banach spaces, *J. Differential Equations* **74** (1988), 285–317.

[Cho-L-L] S.-N. Chow, X.-B. Lin, and K. Lu, Smooth invariant foliations in infinite dimensional spaces, *J. Differential Equations* **94** (1991), 266–291.

[Da-K] D. Daners and P. Koch Medina, "Abstract Evolution Equations, Periodic Problems and Applications, Longman, Harlow," 1992.

[Gi-N-N] B. Gidas, W. Ni, and L. Nirenberg, Symmetry and related properties by the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[Go-S] M. Golubitsky and D. G. Schaeffer, "Singularities and Groups in Bifurcation Theory I," Appl. Math. Sci., Vol. 51, Springer-Verlag, New York, 1986.

[Ha] J. K. Hale, "Asymptotic Behavior of Dissipative Systems," Amer. Math. Soc., Providence, RI, 1988.

[Ha-M-O] J. K. Hale, L. T. Magalhães, and W. M. Oliva, "An Introduction to Infinite-Dimensional Dynamical Systems—Geometric Theory," Springer, New York, 1984.

[He1] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Springer, New York.

[He2] D. Henry, "Perturbation of the Boundary for Boundary Value Problems of Partial Differential Operators," Cambridge Univ. Press, Cambridge, UK, to appear.

[Ka] T. Kato, "Perturbation Theory for Linear Operators," Springer, Berlin, 1966.

[Kr] M. A. Krasnoselskii, "Positive Solutions of Operator Equations," Noordhoff, Groningen, 1964.

[Lu] A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkhäuser, Berlin, 1995.

[Pa-M] J. Palis and W. de Melo, "Geometric Theory of Dynamical Systems," Springer, New York, 1982.

[Pe] J. Peetre, On Hadamard's variational formula, *J. Differential Equations* **36** (1980), 335–346.

[Po1] P. Poláčik, Transversal and nontransversal intersections of stable and unstable manifolds in reaction diffusion equations on symmetric domains, *Differential Integral Equations* **7** (1994), 1527–1545.

[Po2] P. Poláčik, Reaction–diffusion equations and realization of gradient vector fields, "International Conference on Differential Equations (Lisboa, 1995)," World Sci. Publishing, River Edge, NJ, 1998, pp. 197–206.

[Ry] K. P. Rybakowski, An abstract approach to smoothness of invariant manifolds, *Appl. Anal.* **49** (1993), 119–150.

[Sma] S. Smale, Differential dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[Smo] J. Smoller, "Shock Waves and Reaction–Diffusion Equations," Springer, New York, 1967.

[Va1] A. Vanderbauwhede, "Local Bifurcation and Symmetry," Research Notes in Mathematics, Vol. 75, Pitman, London, 1982.

[Va2] A. Vanderbauwhede, Center manifolds, normal forms and elementary bifurcations, *Dynam. Report.* **2** (1989), 89–169.

[Va-I] A. Vanderbauwhede and G. Iooss, Center manifold theory in infinite dimensions, *Dynam. Report. Expositions Dynam. Systems (N.S.)* **1** (1992), 125–163.